Week 1: Algebra

Instructions: This handout is meant to go over some basic algebra techniques and skills. The problems covered here are meant to be of the same level as the NMTC descriptive questions and above that. Problems marked at the end with (*) are a bit more tough (and fun!).
The problems covered here are meant to build your problem-solving skills, and as such, they are open-book. You are encouraged to use any textbook, video, or other online or offline resource as long as it doesn't give away the complete solution.
For each section, there are some worked examples. These examples are meant to show the applications of various techniques and how to approach more difficult problems. It is recommended that you try to attempt all the example problems before reading the solution.

1. Algebraic Manipulation

This section is mainly concerned with system of equations with various variables and how to work with them. Techniques here include algebraic manipulation, factorization, various substitutions and the trivial inequality.
Some useful things to be aware of are:

1. For n odd, $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1})$
2. For integral n, $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$
3. $(a+b+c)(ab+bc+ca) = (a+b)(b+c)(c+a) + abc$
4. (Trivial Inequality) For any real x, $x^2 \geq 0$ with equality if, and only if, $x = 0$

Exmaple 1.1 (Sophie Germain identity). For any x, y real numbers we have that:

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$$

Solution. We proceed by finding a difference of squares. We have:

$$\begin{aligned} x^4 + 4y^4 &= x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 \\ &= (x^2 + 2y^2)^2 - (2xy)^2 \\ &= (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2) \end{aligned}$$

Another example including factorization is as follows:

Example 1.2 For what natural n does the number $n^4 - 12n^2 + 16$ have exactly 2 positive divisors?

Solution. This is another difference of squares. We have:

$$\begin{aligned} n^4 - 12n^2 + 16 &= (n^4 - 8n^2 + 16) - 4n^2 \\ &= (n^2 - 4)^2 - (2n)^2 \\ &= (n^2 - 2n - 4)(n^2 + 2n - 4) \end{aligned}$$

We can re-write this expression as $((n-1)^2 - 5)((n+1)^2 - 5)$. For the expression to only have 2 positive divisors, we must have $(n \pm 1)^2 - 5 = \pm 1$. We get that only $n = \pm 1, \pm 3$ work.

Example 1.3 For real numbers a, b,c, show that if $a^2+b^2+c^2 = ab+bc+ca$ then $a = b = c$

Solution. We re-arrange to get:

$$\begin{aligned}\frac{1}{2}\left((a^2-2ab+b^2)+(b^2-2cb+c^2)+(c^2-2ca+a^2)\right) &= 0\\ (a-b)^2+(c-b)^2+(c-a)^2 &= 0\end{aligned}$$

By the trivial inequality we have that $a-b=0$, $c-b=0$, ad $c-a=0$. This gives us $a=b=c$, as needed.

Example 1.4 Factorize the polynomial:

$$a^2c+b^2a+c^2b-b^2c-a^2b-c^2a$$

Solution. We start by trying to guess a couple factors. Notice that if a term is a factor of the polynomial then setting that factor equal to zero will set the polynomial equal to zero and vice-versa.
It is natural to check if $a-b$ is a factor. Substituting $a=b$ into the polynomial equation, we find that indeed it is a factor. Similarly, we find that $b-c$ and $c-a$ are also factors.
We can verify that the above polynomial factorizes to the expression $(a-b)(b-c)(c-a)$.
Pretty simple so far, so here's a more interesting example:

Example 1.5 (EGMO 2019/1) Find all triples (a, b, c) of real numbers such that $ab+bc+ca=1$ and

$$a^2b+c=b^2c+a=c^2a+b$$

Solution 1. There are multiple ways to approach this. One of them is as follows:
We have that $a^2b+c=b^2c+a$. Ideally, we want to get an isolated ab or bc term so that we can equate the other 2 pairs of equations to get something that gives us $ab+bc+ca$ so that we can use the given condition.

After some trial and error, we manage to get:

$$\begin{aligned} a^2b + c = b^2c + a &\iff a^2b - a = b^2c - c \\ &\iff a(ab-1) = c(b^2-1) \\ &\iff \frac{ab-1}{b^2-1} = \frac{c}{a} \end{aligned}$$

In order to use this, we assume that none of the variables are equal to $\{-1, 0, 1\}$. Observe that if we equate the other equations in the same way, then multiply them, the terms in the RHS will simplify to equal 1. Hence, we do so to get:

$$\frac{(ab-1)(bc-1)(ca-1)}{(a^2-1)(b^2-1)(c^2-1)} = \frac{c}{a} \cdot \frac{b}{c} \cdot \frac{a}{b} = 1$$

Now, use the above expression to get:

$$a^2 + b^2 + c^2 = 1$$

Now we have that $ab + bc + ca = a^2 + b^2 + c^2$, does this look familiar?
Use this to conclude that $a = b = c = \pm\frac{\sqrt{3}}{3}$.
Are these the only solutions? Well, we assumed that none of our terms are equal to $\{-1, 0, 1\}$. So we now have to check when some of our terms are. This is left as an exercise to the reader. In total, there are 8 distinct solutions.

Solution 2: Another way of approaching this problem becomes clear if one is familiar with homogeneous equations. An homogeneous equation is one where each term of the equation is of the same degree. For example. $ab + a^2 = cb$ is an homogenized equation but $ab + a = a^2$ is not since the a term has degree 1 while the other terms have degree 2
This is usefull to us because for an homogenized equation, we can scale all the variables by any non-zero constant value and the equation still holds. To be precise, if $\{a, b, c\}$ is a solution to some homogenized equation, then so is $\{ka, kb, kc\}$ for any non-zero constant k. Hence, if we can homogenize the given equations in the problem, then the additional condition of $ab+bc+ca = 1$ will be automatically taken into account. We homogenize by making the following substitution:

$$a^2b + c(ab + bc + ca) = b^2c + a(ab + bc + ca) = c^2a + b(ab + bc + ca)$$

Notice that after doing this, the condition $ab + bc + ca = 1$ is accounted for as we can now scale all our variables without any issue.
We now get the equation:

$$\begin{aligned} a^2b + c(ab+bc+ca) &= b^2c + a(ab+bc+ca) \\ \iff c^2b + c^2a &= cb^2 + ca^2 \\ \iff c(ac+bc) &= c(b^2+a^2) \end{aligned}$$

If we now once again assume that all our variables are non-zero, we can divide by c to get $ac + bc = b^2 + a^2$. We can equate the other 2 pairs of equations in a similar fashion.
I'm kinda lazy so it is left as an exercise to complete the solution :)
Note: The idea of homogenizing an equation is quite useful and will show up again in various inequalities.

Problems

The problems are ordered roughly in order of difficulty though it may or may not be accurate. You can ask us for hints in the gc or ask for more problems if you aren't satisfied with the ones provided! The last few problems are a bit tough!

Problem 1.1 Find all triples of real numbers a, b, c such that:

$$\begin{aligned} a + b &= a^2 + c^2 \\ b + c &= b^2 + a^2 \\ c + a &= c^2 + b^2 \end{aligned}$$

Problem 1.2 (2025 BMO1 Q2) Find all real numbers x, y, and z such that:

$$\begin{aligned} x^2 + 2yz &= 4 \\ y^2 + 2zx &= 4 \\ z^2 + 2xy &= 1 \end{aligned}$$

Problem 1.3 (2026 Kazakhstan) Let a and b be distinct real numbers such that

$$\frac{a}{b}+\frac{a+10b}{b+10a}=2$$

Find $\dfrac{a}{b}$

Problem 1.4 Let a, b, c be distinct nonzero real numbers such that

$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}.$$

Prove that $|abc|=1$

Problem 1.5 Find all real numbers x, y, w, z such that:

$$\begin{gathered} x+y+z=w \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{w} \end{gathered}$$

Problem 1.6 (OTIS) Factorise the polynomial:

$$a(b-c)^3+b(c-a)^3+c(a-b)^3$$

Problem 1.7 (CMIMC 2020 A7) Let x be a real number, solve the equation:

$$(x-1)(x-4)(x-2)(x-8)(x-5)(x-7)=-48\sqrt{3}$$

Problem 1.8 (Peru MO 2025 L3 P3) Determine all solutions (x, y, z) of the following system of equations:

$$\begin{gathered} \frac{y}{z}=3-2x \\ \frac{z}{x}=3-2y \\ \frac{x}{y}=3-2z \end{gathered}$$

assuming that x, y, and z are non-zero and xyz is an integer.

Problem 1.9 (2010 AIME P9) Find the maximum possible value of $a^3 + b^3 + c^3$ for real numbers a, b, c satisfying:

$$\begin{aligned} a^3 &= abc + 2 \\ b^3 &= abc + 6 \\ c^3 &= abc + 20 \end{aligned}$$

Problem 1.10 (2024 Pakistan TSTST P3) Find all ordered triples of real numbers $(x,\, y,\, z)$ satisfying:

$$\begin{aligned} xyz &= 1 \\ x^2z - 2xz + 2z + xy &= 2 \end{aligned}$$

Problem 1.11 (2017 Indonesia MO P4) Determine all pairs of distinct real numbers (x, y) such that both of the following are true:

$$\begin{aligned} x^{100} - y^{100} &= 2^{99}(x - y) \\ x^{200} - y^{200} &= 2^{199}(x - y) \end{aligned}$$

2. Sequences and Sums ft. Induction

This section will go over sequences, series, and will cover some of the uses of mathematical induction. I'm going to assume that the reader is somewhat familiar with the mentioned things and knows about arithmetic and geometric sequences.

Observe that whenever we have a sum of the form

$$\sum_{k=1}^{n} f(k+1) - f(k)$$

We have that:

$$\sum_{k=1}^{n} f(k+1) - f(k) = f(2) - f(1) + f(3) - f(2) + \cdots + f(n+1) - f(n)$$

Observe that most of the terms cancel and we just get $f(n+1) - f(1)$ remaining. This is an example of a telescopic sum. Here is an example of how they can be used:

Example 2.1 Evaluate the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}$$

Solution. Note that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

We then have:

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} \\ &= 1 - \frac{1}{n+1} \\ &= \frac{n}{n+1} \end{aligned}$$

Example 2.2 (2006 Kazakhstan) Evaluate the sum

$$\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \cdots + \frac{100}{1+100^2+100^4}$$

Solution. Each term in the sum is of the form $\frac{k}{1+k^2+k^4}$. So, can we split this fraction into simpler ones?
Yes, note that $1+k^2+k^4 = (k^2+1)^2 - k^2 = (k^2-k+1)(k^2+k+1)$. From here we get:

$$\frac{k}{(k^2-k+1)(k^2+k+1)} = \frac{1}{2}\left(\frac{1}{k^2-k+1} - \frac{1}{k^2+k+1}\right)$$

Now, since $k^2-k+1 = (k-1)^2 + (k-1) + 1$, this is indeed a telescopic sum. Hence, it simplifies to:

$$\frac{1}{2}\left(1 - \frac{1}{10101}\right) = \frac{5050}{10101}$$

Though, in general, most sums won't die to a well-known trick. Here's an example of a sum(A series actually) that uses somewhat clever algebraic manipulation

Example 2.3 Prove that:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2},$$

provided that $|x| < 1$

Solution. Given the sum $1 + 2x + 3x^2 + \cdots + nx^{n-1}$, we can write it as a sum of the following:

$$\begin{aligned} 1 + x + x^2 + x^3 + \cdots + x^n &= \left(\frac{1-x^{n+1}}{1-x}\right) \\ x + x^2 + x^3 + \cdots + x^n &= x\left(\frac{1-x^n}{1-x}\right) \\ \vdots \quad &\quad \vdots \\ x^n &= x^n\left(\frac{1-x}{1-x}\right) \end{aligned}$$

Adding these sums, we get

$$\begin{aligned} 1 + 2x + 3x^2 + \cdots + nx^{n-1} &= \frac{(1-x^{n+1}) + x(1-x^n) + \cdots + x^n(1-x)}{1-x} \\ &= \frac{1 + x + \cdots + x^n - nx^{n+1}}{1-x} \\ &= \frac{\frac{1-x^{n+1}}{1-x} - nx^{n+1}}{1-x} \end{aligned}$$

Since $|x| < 1$, as $n \to \infty$, $x^n \to 0$.
Hence, $1 + 2x + 3x^2 + \cdots = \dfrac{1}{(1-x)^2}$
Note: Usually you'd use limits to make this argument more rigorous.

Alternate Solution. Notice how the nx^{n-1} term looks an awful lot like a derivative. Indeed, we can write our original sum as

$$\begin{aligned}\sum_{n=1}^{\infty} nx^{n-1} &= \sum_{n=1}^{\infty} \frac{d}{dx}(x^n) \\ &= \frac{d}{dx}\sum_{n=1}^{\infty} x^n \\ &= \frac{d}{dx}\sum_{n=0}^{\infty} x^n \\ &= \frac{d}{dx}(\frac{1}{1-x}) \\ &= \frac{1}{1-x^2}\end{aligned}$$

In the third line, you may note that we changed the index from $n = 1$ to $n = 0$. This doesn't change our final answer because the $n = 0$ term is a constant term, and adding a constant term inside the derivative does not change the derivative.
Had we not shifted the index, the value of our sum would have been $\frac{x}{1-x}$, and the derivative of this is easily verified to be the same as the answer above.

Now, moving on to sequences. Some techniques which could be useful here include algebraic manipulation and various substitutions, including into other sequences. Induction and the method of contradiction can also be used to show something regarding a sequence.

Example 2.4 The sequence $a_0, a_1, \cdots$ is defined as $a_0 = 0$, $a_1 = 1$ and, for $m \geq n \geq 0$, $a_{m+n} + a_{m-n} = \frac{a_{2m} + a_{2n}}{2}$. Prove that for any natural number k, $k \mid a_k$

Solution. We start by trying to analyze this sequence. Setting $n = 0$ gives us $a_{2m} = 4a_m$. This gives us that $a_2 = 4(1)$, $a_4 = 4(4) = 16$, and so on. We have that for any natural k, $a_{2^k} = 2^{2k}$.

This suggests that $a_k = k^2$, so we try to use induction to prove this. The base case is clear so now for the inductive step, we assume that $a_k = k^2$, for all $1 \leq k \leq m$:

$$\begin{aligned} a_{m+1} + a_{m-1} &= \frac{a_{2m} + a_2}{2} = \frac{4a_m + 4}{2} \\ \iff a_{m+1} + (m-1)^2 &= 2m^2 + 2 \end{aligned}$$

It follows that, indeed $a_{m+1} = (m+1)^2$ from which it follows that $k \mid a_k$, for all $k \in \mathbb{N}$.
Note: Normally, we assume that the inductive hypothesis holds for a certain value m, and prove it holds for $m+1$. In this case, we assumed the inductive hypothesis held *for all* values up till m. This is known as *strong induction.*

Finally, we attempt a more challenging problem:

Example 2.5 (2006 IMOSL A2) The sequence of real numbers a_0, a_1, $\cdots$ is defined recursively by

$$a_0 = -1, \qquad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \quad \text{for } n \geq 1$$

Show that $a_n > 0$ for $n \geq 1$.

Solution. If we try to find the starting few values of the sequence, we'll find that there's no obvious pattern to be analyzed. Instead, we try writing down some of the equations in hopes of finding a relation which works well with induction:

$$a_n + \frac{a_{n-1}}{2} + \frac{a_{n-2}}{3} + \cdots + \frac{a_0}{n+1} = 0 \tag{1}$$

$$a_{n-1} + \frac{a_{n-2}}{2} + \frac{a_{n-3}}{3} + \cdots + \frac{a_0}{n} = 0 \tag{2}$$

Where $n > 1$ is a natural number.What if we try subtracting these equations? equations $(2) - (1)$ gives us:

$$\begin{aligned} a_n + a_{n-1}\left(\frac{1}{2} - 1\right) + a_{n-2}\left(\frac{1}{3} - \frac{1}{2}\right) + \cdots + a_0\left(\frac{1}{n+1} - \frac{1}{n}\right) &= 0 \\ \iff a_n - \frac{a_{n-1}}{1 \cdot 2} - \frac{a_{n-2}}{2 \cdot 3} - \cdots - \frac{a_0}{n \cdot (n+1)} &= 0 \\ \iff \frac{a_{n-1}}{1 \cdot 2} + \frac{a_{n-2}}{2 \cdot 3} + \cdots + \frac{a_0}{n \cdot (n+1)} &= a_n \qquad (3) \end{aligned}$$

(3) gives us an expression of a_n in terms of $a_0, a_1, \cdots a_{n-1}$. What other equations uses these terms? Well.. equation (2) does.

Now we realize that if we can show that the LHS of (3) is strictly greater than the LHS of (2), we'll be done. The key idea now is to realize that since the LHS of (2) is equal to 0, we can scale the LHS by any constant that we want. A natural choice is to scale it by $\frac{1}{n+1}$ in order to make the a_0 term disappear when we equate the two equations.

Now we try strong induction. Let's assume that $a_1,\ a_2,\ \cdots,\ a_{n-1} > 0$. We now have to show that:

$$\frac{a_{n-1}}{1 \cdot 2} + \frac{a_{n-2}}{2 \cdot 3} + \cdots + \frac{a_0}{n \cdot (n+1)} > \frac{1}{n+1}\left(a_{n-1} + \frac{a_{n-2}}{2} + \frac{a_{n-3}}{3} + \cdots + \frac{a_0}{n}\right)$$

The pesky and negative a_0 term vanishes and now we have to show:

$$\sum_{k=1}^{n-1} \frac{a_{n-k}}{k(k+1)} > \sum_{k=1}^{n-1} \frac{a_{n-k}}{k(n+1)}$$

By the inductive hypothesis, $a_1, a_2, \cdots a_{n-1} > 0$. So, it suffices to just prove that the following holds for $1 \le k \le n-1$

$$\frac{1}{k(k+1)} > \frac{1}{k(n+1)}$$

But this is easy to verify, and we're done.

Note: The above is usually not how a 'real' proof is written. In contests, you'll usually see all of the motivation to the solution hidden away, leaving only a concise argument.

Note 2: While solving this, I found the bound $a_n \le \frac{n-1}{2n(n+1)}$ for $n \ge 2$. Try to prove this, and determine when equality occurs. Furthermore, try to find a more strict bound and tell it to me if you do!

Problems

The problems are ordered roughly in order of difficulty though it may or may not be accurate. You can ask us for hints in the gc or ask for more problems if you aren't satisfied with the ones provided! The last few problems are a bit tough!
We'll use the notation of $(a_n)_{n\in\mathbb{N}}$, where $\mathbb{N}$ denotes the set of natural numbers to define the sequence a_1, a_2, $\cdots$. Similarly we'll use the notation $(a_n)_{n\in\mathbb{Z}_0^+}$ where $\mathbb{Z}_0^+$ denotes the set of non-negative integers to define the sequence a_0, a_1, $\cdots$

Problem 2.1 Calculate the following sums:

(a) $\displaystyle\sum_{k=1}^{n} k!\cdot k$ (1969 Canada)

(b) $\displaystyle\sum_{k=1}^{n} \frac{k}{(k+1)!}$

(c) $\displaystyle\sum_{k=1}^{n} \frac{k+1}{(k-1)!+k!+(k+1)!}$

(d) $\displaystyle\sum_{k=0}^{\infty} \frac{2^{2^k}}{4^{2^k}-1}$ (IMC 2015)

Problem 2.2 Evaluate

$$\sum_{k=0}^{n} \frac{1}{(n-k)!(n+k)!}$$

Problem 2.3 (Source on AoPs) Evaluate the following sum:

$$\sum_{k=0}^{n} kx^{n-k}$$

Problem 2.4 (2018 SEAMO) Evaluate the following series:

$$\frac{1}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}+...+\frac{n^2}{2^n}+...$$

Problem 2.5 The sequence $(a_n)_{n\in\mathbb{Z}_0^+}$ is defined by $a_0 = 0$, $a_1 = 1$ and for any integers m, $n \geq 0$, by $a_{m+n} + a_m = a_n + a_{2m}$. Find the value of a_{1434}

Problem 2.6 (2009 Croatia) The sequence $(a_n)_{n\in\mathbb{N}}$ is defined by $a_1 = 1$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$, for $n \geq 3$. Prove that $a_n < \left(\frac{7}{4}\right)^n$

Problem 2.7 (Source on AoPs) The sequence $(a_n)_{n\in\mathbb{Z}_0^+}$ is defined by $a_0 = a_1 = 1$ and $a_{n+2} = a_{n+1} - \dfrac{a_n^2}{3}$ for natural number n. Show that $a_n > 0$ for all $n \in \mathbb{N}$

Problem 2.8 Show that the sequence $(a_n)_{n\in\mathbb{N}}$, where $\mathbb{N}$ defined by $0 < a_1 < \frac{1}{2}$ and $a_{n+1} = 2a_n(1 - a_n)$ is strictly increasing. That is, show that $a_n < a_{n+1}$ always holds for any natural number n.
Hint: You may need to use the inequality $\left(\dfrac{x+y}{2}\right)^2 \geq xy$ for positive reals x and y with equality only if $x = y$

Problem 2.9 (2025 Pakistan TST P3)(*) Let $(a_n)_{n\in\mathbb{Z}_0^+}$ be an infinite sequence of real numbers satisfying $\dfrac{a_{n-1} + a_{n+1}}{2} \geq a_n$ for all positive integers n. Show that, for all positive integers n,

$$\frac{a_0 + a_{n+1}}{2} \geq \frac{a_0 + a_1 + \cdots + a_n}{n}$$

Problem 2.10 (2010 BMOSL A2)(*) Let the sequence $(a_n)_{n\in\mathbb{N}}$, where $\mathbb{N}$ denote the set of natural numbers, is given with $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$. Find the minimum real number L, such that for every $k \in \mathbb{N}$

$$\sum_{i=1}^{k} \frac{1}{a_i} < L$$

Problem 2.11 (2014 IMO P1)(*) Let $(a_n)_{n\in\mathbb{N}}$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$